2.6: Properties of Determinants

To each square matrix A, we can associate a scalar called the **determinant** of A and denoted $\det(A)$. The process for computing $\det(A)$ will be described in class.

Theorem 2.64: Let $A \in M_n(\mathbb{R})$.

1. If A has a row or column of zeros, then $\det(A) = 0$.
2. If $A = (a_{ij})$ is an upper triangular, lower triangular, or diagonal matrix, $\det(A) = a_{11}a_{22}\cdots a_{nn}$.
3. For all $n \geq 1$, $\det(I_n) = 1$.
4. For all $B \in M_n(\mathbb{R})$, $\det(AB) = \det(A)\det(B)$.
5. If A is invertible, then $\det(A) \neq 0$ and $\det(A^{-1}) = 1/\det(A)$.

Note: For $A, B \in M_n(\mathbb{R})$, the quantity $\det(A + B)$ does not simplify in any meaningful way. In general, it is not equal to $\det(A) + \det(B)$.

2.7: Rank and Nullity

Let A be any $m \times n$ matrix.

1. The **rank** of A is defined to be the number of nonzero rows in $\text{rref}\,(A)$. (This is equal to the number of leading 1s in $\text{rref}\,(A)$.
2. The **nullity** of A is defined to be the number of free columns in $\text{rref}\,(A)$.
3. $\text{rank}(A) + \text{null}(A) = n$. (Theorem 2.67)
4. If A and B are row-equivalent matrices, then $\text{rank}(A) = \text{rank}(B)$ and $\text{null}(A) = \text{null}(B)$. (Theorem 2.68)

Theorem 2.69: Let A be any $n \times n$ matrix. The following conditions on A are equivalent:

1. $\text{rank}(A) = n$.
2. $\text{null}(A) = 0$.
3. $\text{rref}\,(A) = I_n$.
4. A can be written as the product of elementary matrices.
5. A is invertible.
6. $A\vec{\mathbf{x}} = \vec{0}$ has only the solution $\vec{\mathbf{x}} = \vec{0}$.
7. $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a unique solution for all $\vec{\mathbf{b}} \in \mathbb{R}^n$.
8. $\det(A) \neq 0$.